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Reparametrization Invariance as Gauge Symmetry

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Reparametrization invariance treated as a gauge symmetry shows some specific peculiarities. We study these peculiarities both from a general point of view and by concrete examples. We consider the canonical treatment of reparametrization-invariant systems in which one fixes the gauge on the classical level by means of time-dependent gauge conditions. In such an approach one can interpret different gauges as different reference frames. We discuss the relation between different gauges and the problem of gauge invariance in this case. Finally, we establish a general structure of reparametrizations and its connection with the zero-Hamiltonian phenomenon.

1. INTRODUCTION

Many physical theories are formulated in so-called reparametrization-invariant (RI) form, for instance, models of pointlike relativistic particles, gravity, and string theory. Formally, reparametrization invariance can be treated as a gauge symmetry. However, this gauge symmetry shows some peculiarities, so that it is natural to separate it into a special class of gauge symmetries. For the same reason one has to be careful when formally applying recipes extracted from the consideration of gauge symmetries of a different nature. In all known examples of finite-dimensional RI systems the Hamiltonian vanishes on the constraint surface (in field theory cases this may be not true) in spite of the fact that explicit forms of the reparametrization transformations in these examples may look different. This issue raises a question: What is the general structure of such transformations and is there a definite relation between such a structure and the zero-Hamiltonian phenomenon? The zero-Hamiltonian phenomenon in RI theories raises another well-known problem: What is the time evolution in this case? This question is

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important for the construction of an adequate quantum theory of gravity. Different aspects of RI systems, especially in the example of general relativity, have been studied in numerous publications. One may mention, for example, the papers of Arnowitt *et al.*,[1] where the problem of the zero-Hamiltonian in general relativity was first deeply discussed, and the relevant papers of Kuchar, Hajicek, Isham, and Hartle.[2]

In the canonical schemes under consideration there exists the possibility to introduce the evolution by means of a time-dependent gauge fixing. Fixing the gauge in such a manner, we get different evolutions depending on the selected gauge. Here we meet a question well known in gauge theories: To what extent does the physical content of a theory depend on the gauge fixing and what is gauge invariance here? There exist, in fact, two essentially different points of view on this problem. According to the first one, which is called the "local" point of view, the gauge fixing of the reparametrization gauge freedom corresponds to a certain choice of the reference frame (RF). At the same time, space–time variables in the Lagrangian have to be identified, namely with the coordinates of the above RF. The reparametrizations relate the description of the system in different RF. Thus, one has to admit that local physical quantities may depend on the choice of the gauge. Another, "nonlocal" point of view assumes that there exists a reparametrization-invariant description. Supporters of this position believe that such a description may be realized if one includes an observer in the frame of the theory. Then the physical quantities do not depend on the choice of the gauge, which fixes the reparametrization freedom, and must commute with the corresponding first-class constraints. Unfortunately, the "nonlocal" point of view remains, in the main, declarative. It seems that a clear and convincing realization is lacking. For an excellent and detailed survey on the subject (and relevant references) see ref. 3.

In the present paper we discuss the above and some other questions related to RI theories both from a general point of view and by specific examples. First we analyze the possibility to fix the reparametrization gauge freedom on the classical level in the Hamiltonian formulation. The corresponding gauge conditions have to depend on time (on space–time in field theory case) essentially. In this case the Dirac bracket formalism[4] has to be modified[5]. We apply such a modified approach to the problem under consideration and derive restrictions on the so-called unitary gauges, in which an effective Hamiltonian exists and controls the physical variable dynamics. We analyze relations between different unitary gauges on both the classical and quantum levels in general and in specific examples. We discuss in this connection the model of a relativistic particle in detail. In particular, an explicit relation between the so-called chronological gauge and the proper-time gauge is presented. We advocate the above-mentioned "local" point of view, and consider several examples where one can compare the RI and non-RI versions of the same theory. Namely, we study a finite-dimensional theory, a field theory in a flat space–time, and a theory of the relativistic particle, all of them both in non-RI and in RI form. Based on the considered examples we formulate an interpretation which in fact supports the "local" point of view and gives a specific treatment for the reparametrization gauge symmetry. In the final part of the paper, which seems more formal and independent of the previous part, we study the general structure of the reparametrizations and its relation to the zero-Hamiltonian phenomenon. As a general result in the theory of gauge systems, we prove that if some global continuous symmetry transformation of an action generates a conserved charge which vanishes on the equation of motion, then such an action obeys a gauge symmetry. Leaning upon the latter statement, we establish the general structure of the reparametrizations (the infinitesimal form), which is responsible for the zero-Hamiltonian phenomenon.

2. INTRODUCING REPARAMETRIZATION INVARIANCE

The action of a pointlike relativistic particle

$$S = \int_0^1 L\, d\tau, \qquad L = -m\sqrt{\dot{x}^2}, \qquad x = (x^\mu),\ \dot{x}^\mu = \frac{dx^\mu}{d\tau}, \qquad \mu = 0, \ldots, D \tag{1}$$

gives us a simple example of an RI theory. It is invariant under reparametrizations $x^\mu(\tau) \to x'^\mu(\tau) = x^\mu(f(\tau))$, where f is an arbitrary function obeying only the following demands: $\dot{f}(\tau) > 0, f(0) = 0, f(1) = 1$. The reparametrizations can be interpreted as gauge transformations (GT) whose infinitesimal form is

$$\delta x^\mu(\tau) = \dot{x}^\mu(\tau)\epsilon(\tau), \qquad \delta L = \frac{d}{d\tau}[\epsilon(t)L] \tag{2}$$

where $\epsilon(\tau)$ is a time-dependent parameter. An equivalent Lagrangian function which is adapted to the $m \to 0$ limit contains an additional variable e and is of the form

$$L = -\frac{\dot{x}^2}{2e} - e\frac{m^2}{2} \tag{3}$$

Here the infinitesimal form of the reparametrizations is

$$\delta x^\mu(\tau) = \dot{x}^\mu(\tau)\epsilon(\tau), \qquad \delta e(\tau) = \dot{e}(\tau)\epsilon(\tau) + e(\tau)\dot{\epsilon}(\tau), \qquad \delta L = \frac{d}{d\tau}[\epsilon(\tau)L] \tag{4}$$

String theory is of the same nature; its action is invariant under the reparametrizations of two variables. Gravity is an example of an RI field theory. The Einstein action

$$S_E = \int \mathscr{L}_E \, d^{D+1}x, \qquad \mathscr{L}_E = -\sqrt{-g}\, R \tag{5}$$

is invariant under general coordinate transformations

$$x^\mu \to x'^\mu = x'^\mu(x), \qquad g_{\mu\nu}(x) \to g'_{\mu\nu}(x), \qquad g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\lambda\sigma}(x)$$

These are in fact reparametrizations of $D + 1$ space-time variables. They can be treated as GT,

$$\delta g_{\mu\nu}(x) = D_\mu \epsilon_\nu(x) + D_\nu \epsilon_\mu(x), \qquad \delta \mathscr{L}_E = \partial_\mu[\epsilon^\mu(x)\, \mathscr{L}_E] \tag{6}$$

where D_μ is a covariant derivative and $\epsilon^\mu(x)$ are GT parameters—arbitrary functions on space–time coordinates.

Any action can be extended to an RI form[6]. Consider, for example, a nonsingular action (similar considerations can be made for any singular Lagrangian as well)

$$S = \int_{t_1}^{t_2} L(\mathbf{x}, \dot{\mathbf{x}}, t)\, dt, \qquad \mathbf{x} = (x^i), \quad i = 1, \ldots, D, \qquad \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} \tag{7}$$

Let us change t to x^0 and then replace the integration variable x^0,

$$x^0 = f(t), \qquad f(t_1) = t_1, \qquad f(t_2) = t_2 \tag{8}$$

Thus, we get

$$S_R = \int_{t_1}^{t_2} L_R(x, \dot{x})dt, \qquad L_R(x, \dot{x}) = L\left(\mathbf{x}, \frac{\dot{\mathbf{x}}}{\dot{x}^0}, x^0\right)\dot{x}^0 \tag{9}$$

If we keep in mind the relations (8), the action (9) is completely equivalent to the initial one (7). On the other hand, one can now treat (9) in a new way, namely, we can forget about (8) and treat x^0 as a new independent variable, so that the total set of variables of the theory is $x = (x^\mu) = (x^0, \mathbf{x})$.

Let us analyze the relation between the theory with the actions (9) and (7), in particular, in the Hamiltonian formulation. For the nonsingular theory (7) one can always solve the equations which define the momenta with respect to all velocities:

$$\boldsymbol{\pi} = \frac{\partial L}{\partial \dot{\mathbf{x}}} \Rightarrow \dot{\mathbf{x}} = \boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\pi}, t), \qquad \boldsymbol{\pi} = (\pi_i) \tag{10}$$

Then the time evolution is generated by the Hamiltonian equations without any constraints,

$$\dot{\eta} = \{\eta, H\}, \qquad \eta = (\mathbf{x}, \boldsymbol{\pi}), \qquad H = \left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\dot{\mathbf{x}} - L\right)\Bigg|_{\dot{\mathbf{x}}=\boldsymbol{\psi}} = H(\mathbf{x}, \boldsymbol{\pi}, t) \tag{11}$$

In the theory with the action S_R there primary constraints appear in the Hamiltonian formulation. Indeed, let $\pi_\mu = (\pi_0, \boldsymbol{\pi})$ be momenta conjugate to x^μ,

$$\pi_0 = \frac{\partial L_R}{\partial \dot{x}^0} = -\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\dot{\mathbf{x}} - L\right)\Bigg|_{\dot{\mathbf{x}}\to\dot{\mathbf{x}}/\dot{x}^0,\, t\to x^0}, \qquad \boldsymbol{\pi} = \frac{\partial L_R}{\partial \dot{\mathbf{x}}} = \frac{\partial L}{\partial \dot{\mathbf{x}}}\Bigg|_{\dot{\mathbf{x}}\to\dot{\mathbf{x}}/\dot{x}^0,\, t\to x^0} \tag{12}$$

From the second equation in (12) [taking into account (10)] we get $\dot{\mathbf{x}} = \dot{x}^0\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\pi}, x^0)$, whereas $\dot{x}^0$ is a primarily inexpressible velocity. Then the first equation (12) [taking into account (11)] appears to be a primary constraint

$$\phi_1 = \pi_0 + H(\mathbf{x}, \boldsymbol{\pi}, x^0) = 0 \tag{13}$$

Constructing the total Hamiltonian $H^{(1)}$ according to the standard procedure,[4,5] we get

$$H^{(1)} = \left(\frac{\partial L_R}{\partial \dot{x}^\mu}\dot{x}^\mu - L_R\right)\Bigg|_{\dot{\mathbf{x}}=\dot{x}^0\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\pi},x^0)} = \lambda\phi_1, \qquad \lambda = \dot{x}^0 \tag{14}$$

Thus, the total Hamiltonian vanishes on the constraint surface (on the equations of motion). No more constraints appear from the consistency conditions. To fix a gauge we have to impose a new constraint $\phi_2 = 0$ so that the matrix $\{\phi_a, \phi_b\}$, $a, b = 1, 2$, is not singular. A natural form of a such a condition is $\phi_2 = x^0 - \varphi(\mathbf{x}, \boldsymbol{\pi}, t) = 0$, where the function $\varphi(\mathbf{x}, \boldsymbol{\pi}, t)$ has an essential t dependence, introduced in the theory, in spite of the fact that the Hamiltonian is zero. The simplest choice of the gauge condition is (we will call such a condition the *chronological gauge*)

$$\phi_2 = x^0 - t = 0 \tag{15}$$

The set of second-class constraints (13), (15) explicitly depends on time. The general method to deal with nonstationary constraints in the canonical formulation and quantization procedure were first proposed in ref. 5. Similar results were then obtained by a geometrical approach in ref. 7. The BRST formulation of the nonstationary constraints case was discussed in ref. 8. Below we briefly recall the treatment of ref. 5 for systems with nonstationary second-class constraints.

Consider a theory with second-class constraints $\phi_a(\eta, t) = 0$ (where $\eta = (x^i, \pi_i)$ are canonical variables) which may explicitly depend on time t. Then the equation of motion of such a system may be written by means of the Dirac brackets if one formally introduces a momentum ϵ conjugate to the time t and defines the Poisson bracket in the extended phase space of canonical variables $(\eta; t, \epsilon)$,

$$\dot{\eta} = \{\eta, H + \epsilon\}_{D(\phi)}, \qquad \phi(\eta, t) = 0 \tag{16}$$

where H is the Hamiltonian of the system and $\{A, B\}_{D(\phi)}$ is the notation for the Dirac brackets with respect to the system of second-class constraints ϕ. The Poisson brackets, wherever encountered, are henceforth understood as in the above-mentioned extended space. The quantization procedure in the "quasi-Schrödinger" picture can be formulated in that case as follows. The variables η of the theory are assigned the operators $\tilde{\eta}$, which satisfy the relations

$$[\tilde{\eta}, \tilde{\eta}'] = i\{\eta, \eta'\}_{D(\phi)}\big|_{\eta=\tilde{\eta}}, \qquad \phi(\tilde{\eta}, t) = 0 \tag{17}$$

and equations of evolution

$$\dot{\tilde{\eta}} = \{\eta, \epsilon\}_{D(\phi)}\Big|_{\eta=\tilde{\eta}} = -\{\eta, \phi_a\}C_{ab}\frac{\partial\phi_b}{\partial t}\Bigg|_{\eta=\tilde{\eta}}, \qquad C_{ac}\{\phi_c, \phi_b\} = \delta_{ab} \tag{18}$$

One can demonstrate that (17) and (18) are consistent. To each physical quantity A given in the Hamiltonian formalism by the function $A(\eta, t)$ there corresponds a "quasi-Schrödinger" operator $\tilde{A}$ by the rule $\tilde{A} = A(\tilde{\eta}, t)$; in the same manner one constructs the quantum Hamiltonian $\tilde{H}$ according to the classical one $H(\eta, t)$. The time evolution of the state vectors Ψ in this picture is determined by the Schrödinger equation with the Hamiltonian $\tilde{H} = H(\tilde{\eta}, t)$. The total time evolution results from the evolution both of the state vectors and one of the operators. It is convenient to analyze such an evolution in the Heisenberg picture, whose operators $\check{\eta}$ are related to the operators $\tilde{\eta}$ as $\check{\eta} = U^{-1}\tilde{\eta}U$, where U is the evolution operator related to the Hamiltonian $\check{H}$. Such operators satisfy the equations

$$\dot{\check{\eta}} = \{\eta, H + \epsilon\}_{D(\phi)}\big|_{\eta=\check{\eta}} \tag{19}$$

$$[\check{\eta}, \check{\eta}'] = i\{\eta, \eta'\}_{D(\phi)}\big|_{\eta=\check{\eta}}, \qquad \phi(\check{\eta}, t) = 0$$

All the relations (19) together may be considered as a prescription for quantization in the Heisenberg picture for theories with nonstationary second-class constraints. The total time evolution is controlled only by the first set of equations (19) since the state vectors do not depend on time in the Heisenberg picture. In the general case such an evolution is not unitary. Suppose, however, that part of the set of second-class constraints consists of supplementary gauge conditions, the choice of which is in our hands. In this case we may try to select these gauge conditions in a special form to obtain unitary evolution. The evolution is unitary if there exists an effective Hamiltonian $H_{\text{eff}}(\eta)$ in the initial phase space of the variables η so that the right side of the equations of motion (16) may be written as

$$\dot{\eta} = \{\eta, H + \epsilon\}_{D(\phi)} = \{\eta, H_{\text{eff}}\}_{D(\phi)} \tag{20}$$

In this case [due to the commutation relations (19)] the quantum operators $\check{\eta}$ obey the equations (we disregard here problems connected with operator ordering)

$$\dot{\check{\eta}} = -i[\check{\eta}, \check{H}_{\text{eff}}], \qquad \check{H}_{\text{eff}} = H_{\text{eff}}(\check{\eta}) \tag{21}$$

The latter allows one to introduce the real Schrödinger picture where operators do not depend on time, but the evolution is controlled by the Schrödinger equation with the Hamiltonian H_{eff}. We may call the gauge conditions which imply the existence of the effective Hamiltonians *unitary gauges*. Remember that in the stationary constraint case all gauge conditions are unitary.[5] As is known,[5] the set of second-class constraints can always be solved explicitly with respect to part of the variables $\eta_* = \Psi(\eta^*)$, $\eta = (\eta_*, \eta^*)$, so that η_* and η^* are sets of pairs of canonically conjugate variables $\eta_* = (q_*, p_*)$, $\eta^* = (q^*, p^*)$. We may call η^* independent variables and η_* dependent ones. In fact, $\eta_* - \Psi(\eta^*) = 0$ is a set of second-class constraints equivalent to $\phi(\eta) = 0$. One can easily demonstrate that it is enough to verify the existence of the effective Hamiltonian [the validity of relation (21)] for the independent variables only. Then the evolution of the dependent variables which is controlled by the constraint equations is also unitary.

In the situation of main interest here, when the Hamiltonian is proportional to the constraints, one can put $H = 0$ in equations (19). Thus, the "quasi-Schrödinger" picture and the Heisenberg one coincide. The time evolution is unitary in this case if the following equations hold:

$$\dot{\eta} = \{\eta, \epsilon\}_{D(\phi)} = -\{\eta, \phi_a\}C_{ab}\frac{\partial\phi_b}{\partial t} = \{\eta, H_{\text{eff}}(\eta)\}_{D(\phi)} \tag{22}$$

Let us analyze the theory (2.9) in the gauge (15) using the above consideration. The matrix $\{\phi_a, \phi_b\}$ is simple in this case: $\{\phi_a, \phi_b\} = \text{antidiag}(-1, 1)$, $C_{ab} = \{\phi_b, \phi_a\}$. The Dirac brackets between the independent variables $\mathbf{x}$, $\boldsymbol{\pi}$ are reduced to the Poisson ones,

$$\{x^i, x^j\}_D = \{\pi_i, \pi_j\}_D = 0, \qquad \{x^i, \pi_j\}_D = \delta^i_j \tag{23}$$

The time evolution of these variables is given by the equations

$$\dot{\mathbf{x}} = -\{\mathbf{x}, \phi_a\} C_{ab} \dot{\phi}_b = \{\mathbf{x}, H\}, \qquad \dot{\boldsymbol{\pi}} = -\{\boldsymbol{\pi}, \phi_a\} C_{ab} \dot{\phi}_b = \{\boldsymbol{\pi}, H\} \tag{24}$$

where H is the Hamiltonian of the theory (7) and at the same time it is the effective Hamiltonian in our definition. This means that in the chronological gauge the dynamics of the original nonsingular theory is reproduced.

Let us consider instead of (15) a more general gauge fixing $\phi_2 = x^0 - \varphi(\mathbf{x}, \boldsymbol{\pi}, t) = 0$. To get conditions on the function φ which make the gauge unitary we restrict ourselves to the free particle case, where H from (11) is $\mathbf{p}^2/2m$. In this case $\{\phi_a, \phi_b\} = \text{antidiag}(-K, K)$, $C_{ab} = K^{-2}\{\phi_b, \phi_a\}$, $K = (1 - (\pi_i/m)\, \partial_i \varphi)$. The nonzero Dirac brackets between the independent variables $\mathbf{x}$, $\boldsymbol{\pi}$ are

$$\{x^i, x^j\}_D = (mK)^{-1} \left(\frac{\partial \varphi}{\partial \pi_i} \pi_j - \frac{\partial \varphi}{\partial \pi_j} \pi_i \right), \qquad \{x^i, \pi_j\}_D = \delta^i_j + (mK)^{-1}\, \pi_i \partial_j \varphi \tag{25}$$

According to (22), these variables obey the following equations:

$$\dot{\mathbf{x}} = -\{\mathbf{x}, \phi_a\} C_{ab} \dot{\phi}_b = (mK)^{-1} \boldsymbol{\pi} \dot{\varphi}, \qquad \dot{\boldsymbol{\pi}} = -\{\boldsymbol{\pi}, \phi_a\} C_{ab} \dot{\phi}_b = 0 \tag{26}$$

On the other hand, if the effective Hamiltonian H_{eff} does exist (unitary gauge), one can write

$$\begin{aligned} \dot{x}^i &= \{x^i, H_{\text{eff}}\}_D = (mK)^{-1} \left(\frac{\partial \varphi}{\partial \pi_i} \pi_j - \frac{\partial \varphi}{\partial \pi_j} \pi_i \right) \\ &\quad \times \partial_j H_{\text{eff}} + [\delta^i_j + (mK)^{-1} \pi_i \partial_j \varphi] \frac{\partial H_{\text{eff}}}{\partial \pi_j} \\ \dot{\pi}_i &= \{\pi_i, H_{\text{eff}}\}_D = -\,(\delta^i_j + (mK)^{-1}\, \pi_j \partial_i \varphi)\, \partial_j H_{\text{eff}} \end{aligned} \tag{27}$$

Comparing (26) with (27), we get the following conditions on H_{eff}:

$$\partial_j H_{\text{eff}} = 0, \qquad \frac{\partial H_{\text{eff}}}{\partial \pi_i} = \pi_i (mK)^{-1} \left(\dot{\varphi} - \frac{\partial H_{\text{eff}}}{\partial \pi_i} \partial_i \varphi \right) \tag{28}$$

The first equation (28) means that H_{eff} does not depend on $\mathbf{x}$ and the second one results in the condition

$$\left(\pi_j \frac{\partial}{\partial \pi_i} - \pi_i \frac{\partial}{\partial \pi_j} \right) H_{\text{eff}} = 0 \tag{29}$$

which means that $H_{\rm eff}$ depends only on π^2. Thus, $H_{\rm eff} = H_{\rm eff}(\pi^2, t)$. Using this information in the second equation (28), we get

$$2m\frac{\partial H_{\rm eff}}{\partial \pi^2} = \dot{\varphi} \tag{30}$$

Thus, $\dot{\varphi}$ is a function on π^2 and t only. That leads to the following structure:

$$\varphi(\mathbf{x}, \pi, \mathbf{t}) = \chi(\mathbf{x}, \pi) + \psi(\pi^2, t) \tag{31}$$

where χ and ψ are arbitrary functions on the indicated arguments. The effective Hamiltonian in this case can be expressed via the function $\psi(\pi^2, t)$ only:

$$H_{\rm eff} = \frac{1}{2m}\int \dot{\psi}(\pi^2, t)\, d\pi^2 \tag{32}$$

As an example of a gauge condition that is nonlinear in time t we consider

$$\phi_2 = x^0 - t - \frac{ma}{2\pi}t^2 = 0 \tag{33}$$

where for simplicity we have selected the one-dimensional case, i.e., the Hamiltonian of the initial nonsingular theory is $H = \pi^2/2m$. The previous consideration is valid in this case; thus, (33) is an unitary gauge which generates an effective Hamiltonian of the form

$$H_{\rm eff} = \frac{\pi^2}{2m} + \pi a t \tag{34}$$

If we suppose that the initial nonsingular action (7) corresponds to a theory in an inertial reference frame, then the chronological gauge (15) returns us to the description in such a frame, whereas the gauge (33) corresponds to the description from the point of view of an accelerating (with acceleration a) frame.

Let us turn to the question of physical quantities in the RI theory under consideration. It is known[4,5] that in conventional gauge theories physical quantities which are defined by functions on the phase space have to commute with first-class constraints on the mass shell (Dirac's criterion). What kind of restriction does this criterion impose on the physical quantities in our case? Due to the constraint (13), the physical quantities which are given by functions on the phase space of variables x^μ, π_μ can always be expressed via functions of the form $A = A(x^0, \eta)$, $\eta = (\mathbf{x}, \pi)$. The condition of commutativity of such functions with the first-class constraint (13) on the mass shell then results in

$$\{A, \phi_1\} = \frac{\partial A}{\partial x^0} + \frac{\partial A}{\partial \eta}\{\eta, H\} \approx 0 \tag{35}$$

Remembering that the equations of motion in the theory under consideration have the form

$$\dot{\eta} = \{\eta, H^{(1)}\} = \lambda\{\eta, H\}, \qquad \dot{x}^0 = \{x^0, H^{(1)}\} = \lambda \tag{36}$$

we may rewrite (35) as

$$\frac{\partial A}{\partial x^0}\dot{x}^0 + \frac{\partial A}{\partial \eta}\dot{\eta} = \frac{dA}{dt} \approx 0 \tag{37}$$

Thus, the Dirac criterion admits as physical functions only those which present integrals of motion. We believe that the RI theory under consideration in the chronological gauge (15) has to coincide with the initial nonsingular theory (7), in which all the functions of the form $A = A(t, \eta)$ are physical. Thus, if one accepts the Dirac criterion, then an essential part of real physical quantities of the initial nonsingular theory (7) is lost and the RI version is not equivalent to the initial theory.

The above consideration looks even more transparent in the case of field theory. Let us consider, for example, a theory of a scalar field in a flat space–time. The action of the theory written in an inertial RF has the form

$$S = \int \mathcal{L} d^{D+1}x = \int \left[\frac{1}{2}\eta^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} + F(\varphi)\right] d^{D+1}x \tag{38}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$, $F(\varphi)$ are some terms independent of the derivatives of φ, and $\varphi_{,\mu} = \partial\varphi/\partial x^\mu$. In (38) let us change x^μ to y^μ and then let us rewrite the integral on the RHS of (38) by doing the substitution $y^\mu = y^\mu(x)$. Thus, we get

$$S_R = \int \mathcal{L}_R d^{D+1}x = \int \left[\frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} + F(\varphi)\right] \sqrt{-g}\, d^{D+1}x \tag{39}$$

where

$$g^{\mu\nu} = a^\mu_\alpha a^\nu_\beta \eta^{\alpha\beta}, \qquad a^\mu_\alpha\, y^\alpha_{,\nu} = \delta^\mu_\nu, \qquad g = \det\|g_{\mu\nu}\| = -e^2, \qquad e = \det\|y^\mu_{,\nu}\| \tag{40}$$

and $g_{\mu\nu}$ is the inverse of $g^{\mu\nu}$. If one treats the y^μ as four new scalar fields, then the theory becomes a gauge one, with the corresponding gauge transformations having the form

$$\delta y^\mu = y^\mu_{,\alpha}\delta\zeta^\alpha, \qquad \delta\varphi = \partial_\alpha\varphi\delta\zeta^\alpha \tag{41}$$

where $\delta\zeta(x)$ are $D + 1$ x-dependent parameters of the gauge transformations. To see the relation between the theories (38) and (39) we construct their Hamiltonian versions as in the previous finite-dimensional case. Let us start with the gauge theory (39). Using the relations

$$\frac{\partial e}{\partial \dot{y}^\alpha} = ea^0_\mu, \qquad \frac{\partial g^{\mu\nu}}{\partial \dot{y}^\alpha} = -2g^{0\mu}a^\nu_\alpha \tag{42}$$

we introduce the canonical momenta:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = ea^0_\mu a^0_\nu g^{\mu\nu}\dot{\varphi} + ea^0_\mu a^i_\nu g^{\mu\nu}\varphi_{,i}$$

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{y}^\mu} = -\frac{1}{2}\, ea^0_\mu\, a^0_\nu a^0_\rho g^{\nu\rho}\dot{\varphi}^2$$

$$-ea^i_\mu a^0_\nu a^0_\rho g^{\nu\rho}\dot{\varphi}\varphi_{,i} + e\left[\frac{1}{2}\, a^0_\mu a^i_\nu a^i_\rho - a^i_\mu a^i_\nu a^0_\rho\right] g^{\nu\rho}\varphi_{,i}\varphi_{,j} + ea^0_\mu F(\varphi) \tag{43}$$

Equations (43) allow one to express only the velocity $\dot{\varphi}$ via fields and momenta; velocities $\dot{y}^\mu$ remain inexpressible,

$$\dot{\varphi} = \frac{\pi - ea^0_\mu a^i_\nu g^{\mu\nu}\,\varphi_{,i}}{ea^0_\mu a^0_\nu g^{\mu\nu}} \tag{44}$$

Thus, the primary constraint $\phi_1 = 0$ appear:

$$\phi_{1\mu} = \pi_\mu + a^\alpha_\mu(y)\mathcal{H}_\alpha(y) \tag{45}$$

where

$$\mathcal{H}_0(y) = \frac{\pi^2}{2eg^{00}} - \frac{g^{0i}}{g^{00}}\,\varphi_{,i}\pi - \frac{e}{2}\frac{\gamma^{ij}}{g^{00}}\,\varphi_{,i}\varphi_{,j} - eF(\varphi) \tag{46}$$

$$\mathcal{H}_i(y) = \varphi_{,i}\pi, \qquad \gamma^{ij} = -\frac{g^{0i}g^{0j}}{g^{00}} + g^{ij} \tag{47}$$

The density of the total Hamiltonian is

$$\mathcal{H}^{(1)} = \lambda^\mu\phi_{1\mu}, \qquad \lambda^\mu = \dot{y}^\mu \tag{48}$$

where the inexpressible velocities $\dot{y}^\mu$ appear as Lagrange multipliers. No more constraints appear and ϕ_1 are first-class constraints. A possible form of the gauge conditions is

$$\phi_2^\mu = y^\mu - f^\mu(x) = 0, \qquad \left|\frac{\partial f}{\partial x}\right| \neq 0 \tag{49}$$

Together with the primary constraints they form a set of second-class constraints, which can be written in the equivalent form $\Phi = 0$, where

$$\Phi = \begin{cases} \pi_\mu + a_\mu^\alpha(f(x))\mathcal{H}_\alpha\,(f(x)) = 0 \\ y^\mu - f^\mu(x) = 0 \end{cases} \tag{50}$$

One can select $Q = (\varphi, \pi)$ as independent variables. The Dirac brackets between them are

$$\begin{aligned} \{\varphi, \pi\}_{D(\Phi)} &= \{\varphi, \pi\} = 1, \\ \{\varphi, \varphi\}_{D(\Phi)} &= \{\varphi, \varphi\} = 0, \\ \{\pi, \pi\}_{D(\varphi)} &= \{\pi, \pi\} = 0 \end{aligned} \tag{51}$$

The time evolution is given by an effective Hamiltonian,

$$\dot{Q} = -\{Q, \Phi_A\}C_{AB}\dot{\Phi}_B = \{Q, H_{\text{eff}}\}, \qquad C_{AB} = \{\Phi, \Phi\}_{AB}^{-1}$$
$$H_{\text{eff}} = \int \mathcal{H}_0(f(x))\,\mathbf{dx} \tag{52}$$

Thus, the gauge (49) is unitary. One can easily see that the equations of motion (52) reproduce the dynamics of the initial theory of the scalar field in flat space, but in a curvilinear RF, the coordinates x of which are related to the coordinates y of the inertial RF by the transformation (49). If $f^\mu(x) = x^\mu$ [an analog of the chronological gauge (15) of the finite-dimensional case] or $f^\mu(x) = \Lambda_\nu^\mu\, x^\nu$ ($\Lambda^T\eta\Lambda = \eta$), then we get back to the initial theory in an inertial RF. In this case the effective Hamiltonian (52) takes the familiar form

$$H_{\text{eff}} = \int \mathcal{H}\mathbf{dx} = \int\left[\frac{1}{2}(\pi^2 + \varphi_{,i}^2) - F(\varphi)\right]\mathbf{dx} \tag{53}$$

What are the physical quantities in the theory (39)? The Dirac criterion admits only those which commute with all first-class constraints. In our case, that would mean

$$\{A, \phi_{1\mu}\} \approx 0 \tag{54}$$

where ϕ_1 is given in (45). Due to the same constraint (45), the physical quantities, which are functions on the phase space, always can be taken in the form $A = A(y, \eta)$, $\eta = (\varphi, \pi)$. For such functions the condition (54) results in

$$\{A, \phi_{1\mu}\} = \frac{\partial A}{\partial y^{\mu}} + \frac{\partial A}{\partial \eta} a_{\mu}^{\alpha} \{\eta, H_{\alpha}\} \approx 0 \tag{55}$$

Multiplying this equation by the nonsingular matrix $y,^{\mu}_{\beta}$ gives the following relation:

$$\frac{dA}{dx^{\mu}} \approx 0 \tag{56}$$

which is a generalization of the finite-dimensional equation (37). Equation (56) means that the above criterion admits as physical only functions that do not depend on space–time.

Similar to the finite-dimensional case, we meet here the following situation. If we accept the Dirac criterion, then we cannot identify the RI version of the scalar field theory with the initial formulation in flat space–time even in the "chronological" gauge. That circumstance indicates that the above criterion has to be critically reconsidered in the situation under consideration (for detailed discussion see the next section).

3. RELATIVISTIC PARTICLE THEORY. RI AND TIME INVERSION

In this section we discuss the theory of a relativistic particle as an instructive example of an RI system. Such a theory is interesting by itself and has attracted attention for a long time, in particular due to the fact that it can serve as a prototype for a string theory (now one can consider it as a 0-brane theory). By this example we are going to study different possibilities of time-dependent gauge fixing and a relation between reparametrizations and time-inversion symmetry.

Let us restrict ourselves for simplicity to spinless particles moving in an external electromagnetic field with the potentials $A^{\mu} = (0, \mathbf{A}(\mathbf{x}))$, which corresponds to the case of a constant magnetic field. The theory of such a particle is described by the action[9]

$$S = \int \left[-m\sqrt{1 - (\dot{\mathbf{x}}^2)} + g\dot{\mathbf{x}}\mathbf{A} \right] dt \tag{57}$$

where $\mathbf{x} = (x^i)$ are spatial coordinates of some inertial reference frame and t is the time of the same frame, g is the algebraic charge of the particle, and m is its mass. The action (57) is nonsingular, so that Hamiltonization and quantization can be done directly. The three-dimensional momentum vector $\boldsymbol{\pi}$ is defined by

$$\boldsymbol{\pi} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{m\dot{\mathbf{x}}}{\sqrt{1 - (\dot{\mathbf{x}}^2)}} + g\mathbf{A}, \qquad \boldsymbol{\pi} = (\pi_i) \tag{58}$$

The classical equations of motion are

$$\dot{\eta} = \{\eta, \omega\}, \qquad \eta = (\mathbf{x}, \boldsymbol{\pi}), \qquad \omega = \sqrt{m^2 + (\boldsymbol{\pi} - g\mathbf{A})^2} \tag{59}$$

They describe the motion of a particle with charge g in a constant magnetic field. Going over to the quantum theory, we get the commutation relations between the operators $\hat{\mathbf{x}}$, $\hat{\boldsymbol{\pi}}$: $[\hat{x}^i, \hat{\pi}_k] = i\{x^i, \pi_k\} = i\delta^i_k$. In the coordinate representation $\hat{\mathbf{x}}$ is a multiplication operator, whereas $\hat{\boldsymbol{\pi}} = -i\,\partial/\partial\mathbf{x}$. The state vectors ψ obey the Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \hat{\omega}\psi, \qquad \hat{\omega} = \sqrt{m^2 + (i\nabla + g\mathbf{A})^2} \tag{60}$$

The quantum theory constructed in this way describes only one particle with charge g. Such a theory is not equivalent to a theory based on the Klein–Gordon equation. Indeed, the latter describes states of charged particles with positive and negative energies, or states of particles and antiparticles [charge $(-g)$] with positive energies.

Let us consider an RI formulation of the system in question. The corresponding action has the form

$$S = \int [-m\sqrt{\dot{x}^2} - g\dot{x}^\mu A_\mu]\, d\tau, \qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau} \tag{61}$$

where now four $x^\mu = (x^0, \mathbf{x})$ are dynamical variables dependent on a new time τ. The action (61), similar to the one (1), obeys the reparametrization gauge symmetry (2). Hamiltonization and quantization of the theory is more complicated than in the previous case. Let π_μ be the generalized momenta related to the variables x^μ,

$$\pi_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -\frac{m\dot{x}_\mu}{\sqrt{\dot{x}^2}} - gA_\mu \tag{62}$$

Then there is a constraint $(\pi + gA)^2 = m^2$, which can be written in the following equivalent form, which is convenient for our purposes:

$$\phi_1 = \pi_0 + \zeta\omega = 0, \qquad \zeta = -\text{sign}\ \pi_0 \tag{63}$$

One can express from (62) three velocities $\dot{\mathbf{x}}$ as well as the sign of $\dot{x}^0$ in terms of the coordinates, momenta, and one inexpressible velocity, which here is $\lambda = |\dot{x}^0|$,

$$\dot{\mathbf{x}} = \lambda\omega^{-1}(\boldsymbol{\pi} - g\mathbf{A}), \qquad \text{sign}\ \dot{x}^0 = \zeta, \qquad \sqrt{\dot{x}^2} = m\lambda\omega^{-1} \tag{64}$$

Thus, one can construct the total Hamiltonian $H^{(1)}$ by substituting (64) in the expression $\pi_\mu \dot{x}^\mu - L$,

$$H^{(1)} = \lambda\zeta\phi_1 \tag{65}$$

where λ is a Lagrange multiplier subject, however, to the condition of positivity. The Hamiltonian equations of motion of the form

$$\dot{x}^\mu = \{x^\mu, H^{(1)}\}, \qquad \dot{\pi}_\mu = \{\pi_\mu, H^{(1)}\}, \qquad \phi_1 = 0, \quad \lambda \geq 0 \tag{66}$$

are equivalent to the Lagrangian ones. No secondary constraints arise from the consistency conditions and λ remains undetermined. This indicates that we are dealing with a gauge theory. The total Hamiltonian is proportional to the constraints, as one can expect for an RI theory. Below we discuss some possible gauges and quantization in these gauges.

First, let us consider the case of a neutral ($g = 0$) particles. In this case the action (61) is invariant under the time inversion $\tau \to -\tau$. Since the gauge symmetry in the case under consideration is related to the invariance of the action under the changes of the variables τ, two possibilities appear, namely, to include or not to include the above discrete symmetry in the gauge group together with continuous reparametrizations. Let us first study the former possibility and include the time inversion in the gauge group. Then the gauge conditions have to fix the gauge freedom which corresponds to both kind of symmetries, namely, to fix the variable $\lambda = |\dot{x}^0|$, which is related to the reparametrizations, and to fix the variable $\zeta = \text{sign}\ \dot{x}^0$, which is related to the time inversion. To this end we may select the chronological gauge of the form

$$\phi_2 = x^0 - \tau = 0 \tag{67}$$

The consistency condition $\dot{\phi}_2 = 0$ leads on the constraint surface to the equation

$$\dot{\phi}_2 = \frac{\partial\phi_2}{\partial\tau} + \{\phi_2, H^{(1)}\} = -1 + \lambda\zeta = 0 \tag{68}$$

which results in the condition $\zeta\lambda = 1$. Remembering that $\lambda \geq 0$, we get $\zeta = 1$, $\lambda = 1$. That reduces the constraint surface to the form $\phi_a = 0$, $a = 1, 2$,

$$\phi_1 = \pi_0 + \omega, \qquad \phi_2 = x^0 - \tau \tag{69}$$

It is easy to calculate that $\{\phi_a, \phi_b\} = \text{antidiag}(-1, 1)$ and $C_{ab} = -\{\phi_a,$

$\phi_b\}$, $C_{ab}\{\phi_b, \phi_c\} = \delta_{ac}$. One can select $\eta = (\mathbf{x}, \pi)$ as independent variables. Their Dirac brackets coincide with the Poisson ones,

$$\{\eta, \eta'\}_D = \{\eta, \eta'\} \tag{70}$$

The quantum operators $\check{\eta}$ obey equation (19), which in this particular case takes the form

$$\dot{\check{\eta}} = -\{\eta, \phi_a\} C_{ab} \left.\frac{\partial \phi_b}{\partial \tau}\right|_{\eta=\check{\eta}} = \{\eta, \omega\}\big|_{\eta=\check{\eta}} = -i[\check{\eta}, \check{\omega}] \tag{71}$$

$$[\check{\eta}, \check{\eta}'] = i\{\eta, \eta'\}$$

Thus, the evolution is unitary and is governed by the effective Hamiltonian ω, (59). One can consider time-independent Schrödinger operators $\hat{\eta} = e^{-i\check{\omega}\tau}\check{\eta}(\tau)e^{i\check{\omega}\tau}$ and time-dependent state vectors. The operators $\hat{\eta}$ obey the same commutation relations (71) and can be realized as in the non-reparametrization-invariant case. Thus, one gets the Schrödinger equation (60) if one identifies τ with t.

Suppose we do not include the time inversion in the gauge group. That is especially natural when $g \neq 0$, $A_\mu \neq 0$ because in this case the time inversion is no longer a symmetry of the action. Thus, one may now consider the more general situation of a charged particle moving in an external magnetic field. Under the above supposition, the condition (67) is no longer a gauge; it fixes not only the reparametrization gauge freedom (fixes λ), but it also fixes the variable ζ, which is now physical. A possible gauge condition has the form[10]

$$\phi_2 = x^0 - \zeta\tau = 0 \tag{72}$$

The consistency condition $\dot{\phi}_2 = 0$ leads to the equation

$$\dot{\phi}_2 = \frac{\partial \phi_2}{\partial \tau} + \{\phi_2, H^{(1)}\} = -\zeta + \lambda\zeta = 0 \tag{73}$$

which fixes only $\lambda = 1$ and retains ζ as a physical variable. Trajectories with $\zeta = +1$ correspond to particles, while trajectories with $\zeta = -1$ to antiparticles.[10] Two second-class constraints

$$\phi_1 = \pi_0 + \zeta\omega, \qquad \phi_2 = x^0 - \zeta\tau \tag{74}$$

form the same algebra as in the previous case. One has only to add the relation $\{\zeta, \eta\}_D = 0$ to the Dirac brackets (70). However, we get here an additional operator $\check{\zeta}$, which has to be realized in the Hilbert space of state vectors. We assume the operator $\check{\zeta}$ to have the eigenvalues $\zeta = \pm 1$ by analogy with the classical theory. Such an operator can be realized in a Hilbert space whose elements are two-component columns

$$\Psi = \begin{pmatrix} \Psi_1(\mathbf{x}) \\ \Psi_2(\mathbf{x}) \end{pmatrix} \tag{75}$$

if we chose the operator $\hat{\zeta}$ as the matrix $\hat{\zeta} = \mathrm{diag}(1, -1)$. The time-independent operators $\hat{\eta}$ can be realized as follows:

$$\hat{x}^i = x^i \mathbf{I}, \qquad \hat{\pi}_j = -i\partial_j \mathbf{I} \tag{76}$$

where $\mathbf{I}$ is a unit 2×2 matrix. The time evolution of the state vectors is described by the Schrödinger equation

$$i \frac{\partial \Psi}{\partial \tau} = \hat{\omega} \Psi \tag{77}$$

where $\hat{\omega}$ is given by (60). Equation (77) differs from the similar equation (60) due to the structure of the Hilbert space, which now allows one to describe states for both particles and antiparticles.

As an example of gauge conditions which lead to the description from the point of view some noninertial reference frames we consider here the gauge (a = const)

$$\phi_2 = x^0 + \frac{\pi_0}{m} \tau + a = 0 \tag{78}$$

in the case when the time inversion is not included in the gauge group and the gauge

$$\phi_2 = x^0 - \frac{|\pi_0|}{m} \tau + a = 0 \tag{79}$$

when it does.

One can demonstrate first that the gauge condition (78) corresponds (at any a) to the proper-time gauge $\dot{x}^2 = 1$ in the Lagrangian formulation. Indeed the consistency condition

$$\dot{\phi}_2 = \frac{\partial \phi_2}{\partial \tau} + \{\phi_2, H^{(1)}\} = \frac{\pi_0}{m} + \lambda\zeta = 0 \tag{80}$$

defines $\lambda = |\pi_0|/m$. Remembering the last relation (64) and the constraint (63), we can see that (78) at any a is equivalent to the condition $\dot{x}^2 = 1$. Thus, (78) may be called the proper-time gauge in the Hamiltonian formulation. The proper-time gauge, similar to the chronological gauge (72), does not fix the variable ζ, and leaves the possibility to describe particles and antiparticles at the same time. The gauge condition (79), similar to (67), fixes the variables ζ; thus it is acceptable only when the time inversion (78) is included in the gauge group.

The constraint algebra in both gauges (72) and (78) is the same, and the commutation relations and the realization for the independent operators are also the same; however, the effective Hamiltonian in the proper-time gauge is different,

$$H_{\mathrm{eff}} = \frac{\omega^2}{2m} \tag{81}$$

Thus, the Schrödinger equation has the form

$$i\frac{\partial \Psi}{\partial \tau} = \frac{\hat{\omega}^2}{2m}\Psi \tag{82}$$

One can establish a formal relation between the gauges (72) and (78). Namely, one can present a canonical transformation which connects both gauges on the classical level. The generating function of a such transformation has the form

$$W = x^{\mu}\pi'_{\mu} + \tau\left|\pi'_0\right| - \tau\frac{\pi_0'^2}{2m} \tag{83}$$

if the phase space variables without the primes are related to the chronological gauge (72) and the primed ones to the proper-time gauge (78). The transformation does not change the variables x^i and π_{μ}. It changes only x^0, $x'^0 = x^0 - \zeta\tau - (\pi_0/m)\,\tau$. Thus, it transforms the constraint surface of the first gauge into the one of the second gauge. One can also see that this transformation connects both Hamiltonians

$$H = H' + \frac{\partial W}{\partial \tau} = \frac{\pi_0'^2}{2m} + \left|\pi'_0\right| - \frac{\pi_0'^2}{2m} = \left|\pi'_0\right| = \left|\pi_0\right| = \omega \tag{84}$$

On the quantum level the state vectors in both gauges are connected by means of a quantum canonical transformation

$$\Psi = e^{-i\hat{W}}\Psi', \qquad \hat{W} = \tau\hat{\omega} - \tau\frac{\hat{\omega}^2}{2m} \tag{85}$$

In the spirit of the interpretation given in Section 3 we may say that the chronological gauges (67) and (72) lead to the inertial RF, whereas the proper-time gauges (78) and (79) correspond to the description from the point of view of noninertial (at $A \neq 0$) RF. A formal possibility to connect these two gauges by means of a canonical transformation does not mean their physical equivalence since such a transformation depends explicitly on time.

4. POSSIBLE INTERPRETATION

Results of the previous sections may be summarized in the following interpretation. Let us turn first to the non-RI actions (2.7), (2.38), and (3.1). It is natural to believe that such actions give descriptions of the corresponding physical systems in certain RF. For example, actions (2.38) and (3.1) provide a description from the point of view of an inertial RF with a Cartesian base. Constructing RI versions of the above-mentioned actions, we see that the possibility appears to describe the same physical system from the point of view of a wider class of RF. The theories become gauge ones, and contain additional nonphysical variables. The corresponding gauge symmetry RI leads always to the zero Hamiltonian phenomenon. To introduce a dynamics we fix a gauge by means of supplementary conditions which depend on time (or space–time variables) explicitly. It turns out that such a gauge fixing looks literally like a certain choice of RF. In particular, the chronological gauges correspond to the RF in which initial non-RI actions are formulated. More complicated gauges reproduce in general noninertial curvilinear RF. Based on experience derived from the simple example consideration we believe that any fixing of the reparametrization gauge freedom always corresponds to a certain choice of the space–time RF. Here we have especially emphasized the origin of the RF which is fixed. The point is that the fixing of the gauge freedom of any kind can be treated as a choice of some RF. In this sense the reparametrization symmetry is similar to gauge symmetries of a different nature, let us call them internal gauge symmetries (one may define the latter symmetries as ones which do not involve the space–time coordinate transformations). The principal distinction between the reparametrizations and internal gauge symmetries is related to the distinction between the corresponding RF. Whereas one believes that the RF for the internal gauge symmetries may not be realized physically (at least until now), the choice of RF to measure space–time coordinates may be physically realized. If in the former case the physical quantities do not depend on the choice of the gauge, in the latter case this may not be true. To describe local physical quantities it is natural to use space-time-dependent functions which depend explicitly on the choice of RF and are transformed in a certain way under the RF change. Thus, we have to admit gauge-noninvariant objects to describe physics. As is known[4,5] when the gauge transformations do not involve a transformation of space–time coordinates, gauge-invariant functions on the phase space have to commute with first-class constraints on the mass shell (Dirac criterion). The previous reasoning means that the "local" point of view, which is in fact advocated here, abrogates the Dirac's criterion with respect to the first-class constraints which generate the reparametrizations. Rejection of the Dirac criterion in the case of the reparametrization gauge symmetry thus admits any functions (which are physical with respect to the internal gauge symmetries) as physical ones. Their choice is dictated by concrete conditions of the problem. Let us, for example, return to the theory of the scalar field studied in Section 2. Let us have a Lorentz tensor in the initial non-RI formulation, say the vector $\varphi_{,\mu}(x)$. The question is: What kind of physical quantity corresponds to it in the RI formulation? One may present two naturally constructed quantities, the general coordinate vector $\varphi_{,\mu}(x)$ and the scalar $a_{\alpha}^{\mu}\varphi_{,\mu}(x)$. Both coincide with the initial physical quantity in the chronological gauge (in the inertial RF). In the literature one often meets arguments in favor of the latter choice (see, for example, ref. 11).

We know that gauges which fix an internal gauge symmetry can always be selected in time (space–time)-independent form (canonical gauges). Such gauges then may be related by means of a time-independent canonical transformation.[5] In such a way, a formal equivalence between descriptions in different gauges may be established. As we have seen from the examples in Sections 2 and 3, the time-dependent gauges in RI theories may also be connected by means of canonical transformations (such a possibility certainly follows from general theorems[5]). However, such transformations necessarily depend on time (space–time variables). Thus, in this case a formal possibility to connect different gauges does not mean their literal physical equivalence. The canonical transformations in such a case establish only a relation between descriptions of one and the same system in different RF.

5. RI IN GENERAL AND THE ZERO-HAMILTONIAN PHENOMENON

Above we have considered several examples of RI systems. The explicit form of the corresponding GT depends on the structure of the theory [compare (2) and (4)]. At the same time, in all known examples the total Hamiltonian vanishes on the constraint surface of the theory. Is it possible to discover some specific structure of RST in general and a relation of the latter with the zero-Hamiltonian phenomenon? Below we discuss this problem and present such a relation.

Let us have a theory with a finite number of degrees of freedom which is described by an action ($q = q^a$, $a = 1, \ldots, D$, are generalized coordinates and t is time)

$$S = \int L(q, \dot{q}, t)\, dt \tag{86}$$

Consider a transformation in the space of trajectories $q^a(t)$,

$$q^a(t) \rightarrow q'^a(t) = G_t^a(q) \tag{87}$$

where $G_t^a(q)$ are functionals on $q^a(t)$ depending parametrically on time. We will call (87) a symmetry transformation (ST) of the theory if the Lagrangian function $L(q, \dot{q}, t)$ is changed under such a transformation only by a total derivative of some function,

$$L'(q, \dot{q}, t) = L(G_t(q), \dot{G}_t(q), t) = L(q, \dot{q}, t) + \frac{dF}{dt} \tag{88}$$

One can see that the Lagrangians $L(q, \dot{q}, t)$ and $L'(q, \dot{q}, t)$ have the same extremals. That can be regarded as an argument in favor of the proposed definition of the ST.

The ST can be discrete, continuous global, or gauge. Continuous global ST are parametrized by a set of parameters ϵ_α, $\alpha = 1, \ldots, r$. It is convenient to define the point $\epsilon_\alpha = 0$ as the one that corresponds to the identical transformation. In this case (87) can be presented in the form

$$q'^a(t) = G_t^a(q|\epsilon), \qquad G_t^a(q|0) = q^a(t) \tag{89}$$

where the ϵ dependence is indicated explicitly. The infinitesimal form of a global continuous ST is

$$q'^a(t) = q^a(t) + \delta q^a(t), \qquad \delta q^a(t) = \rho_\alpha^a(t)\epsilon_\alpha, \qquad \rho_\alpha^a(t) = \left.\frac{\partial G_t^a(q|\epsilon)}{\partial \epsilon_\alpha}\right|_{\epsilon=0} \tag{90}$$

where $\rho_\alpha^a(t)$ are the generators of the transformations. Continuous ST are GT (or local ST) if they are parametrized by some arbitrary functions of time (or in the case of field theories by functions of space–time variables). They can be presented in the form (89), where, however, $G_t^a(q|\epsilon)$ may depend not only on ϵ, but on its derivatives over time. In this case

$$\delta q^a(t) = \int R_\alpha^a(t, t')\epsilon_\alpha(t')\, dt', \qquad R_\alpha^a(t, t') = \left.\frac{\delta G_t^a(q|\epsilon)}{\delta \epsilon_\alpha(t')}\right|_{\epsilon=0} \tag{91}$$

As it was demonstrated in ref. 5, the generators $R_\alpha^a(t, t')$ are local in time (in the case of ordinary bosonic variables), i.e., they have the following structure:

$$R_\alpha^a(t, t') = \sum_{k=0}^{M} \rho_{\alpha(k)}^a(t)\partial_t^k\, \delta(t - t') \tag{92}$$

where M is finite. Thus, one can write in this case

$$\delta q^a(t) = \sum_{k=0}^{M} \rho^a_{\alpha(k)}(t)\epsilon^{(k)}_\alpha(t), \qquad \epsilon^{(k)}_\alpha(t) = \frac{d^k\epsilon_a(t)}{dt^k} \tag{93}$$

The presence of the r-parameter continuous global ST indicates that there exist r conserved charges. Indeed, in this case $\delta L = (d/dt)\,\delta F$, which is an infinitesimal form of (88). The variations δL and δF can be represented as follows:

$$\delta L = \frac{\delta S}{\delta q^a}\delta q^a + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q^a\right) = \left[\frac{\delta S}{\delta q^a}\rho^a_\alpha + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\rho^a_\alpha\right)\right]\epsilon_\alpha, \qquad \delta F = f_\alpha\epsilon_\alpha \tag{94}$$

where

$$\frac{\delta S}{\delta q^a} = \frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right)$$

so that $\delta S/\delta q^a = 0$ are the Euler–Lagrange equations of motion. Thus, we get

$$\frac{dQ_\alpha}{dt} = -\rho^a_\alpha\frac{\delta S}{\delta q^a}, \qquad Q_\alpha = \frac{\partial L}{\partial \dot{q}^a}\rho^a_\alpha - f_\alpha \tag{95}$$

and therefore Q_α are the above-mentioned conserved charges. An analogous statement is valid for GT as well. Moreover, in this case one can make some conclusions about the structure of the corresponding conserved charges. Below we formulate and prove some statements which are useful for our purposes.

Let an action obey a gauge ST. In the infinitesimal form this results in the condition

$$\delta L = \frac{\delta S}{\delta q^a}\delta q^a + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\delta q^a\right) = \frac{d}{dt}\delta F \tag{96}$$

where δq^a are given by (93) and δF is a function. Similarly to the derivation of (94), (95), this implies the conservation law

$$\frac{dQ}{dt} = -\frac{\delta S}{\delta q^a}\delta q^a, \qquad Q = \left(\frac{\partial L}{\partial \dot{q}^a}\delta q^a - \delta F\right) \tag{97}$$

The conserved charge Q may be represented in the form

$$Q = \sum_{k=0}^{M'} Q_{\alpha(k)}(t)\epsilon^{(k)}_\alpha(t) \tag{98}$$

Substituting (93) and (98) into (97), we get

$$\sum_{k=0}^{M'} [\dot{Q}_{\alpha(k)}\epsilon_\alpha^{(k)}(t) + Q_{\alpha(k)}\epsilon_\alpha^{(k+1)}(t)] = -\frac{\delta S}{\delta q^a}\sum_{k=0}^{M} \rho^a_{\alpha(k)}\epsilon_\alpha^{(k)}(t) \tag{99}$$

It is clear that $M' = M - 1$. Due to the arbitrariness of $\epsilon_\alpha(t)$, one can consider the derivatives $\epsilon_\alpha^{(k)}(t)$ as independent arbitrary functions and compare the terms on the left- and right-hand sides of (99) with the same $\epsilon_\alpha^{(k)}(t)$. Thus one gets

$$Q_{\alpha(M-1)} = -\frac{\delta S}{\delta q^a}\rho^a_{\alpha(M)}, \qquad \dot{Q}_{\alpha(M-1)} + Q_{\alpha(M-2)} = -\frac{\delta S}{\delta q^a}\rho^a_{\alpha(M-1)}, \ldots$$

$$\dot{Q}_{\alpha(k)} + Q_{\alpha(k-1)} = -\frac{\delta S}{\delta q^a}\rho^a_{\alpha(k)}, \ldots \tag{100}$$

It follows from the system (100) that

$$Q_{\alpha(k)} = \Lambda^a_{\alpha(k)}\frac{\delta S}{\delta q^a}, \qquad \text{or} \qquad Q = \Lambda^a\frac{\delta S}{\delta q^a}, \qquad \Lambda^a = \sum_{k=0}^{M-1}\epsilon_\alpha^{(k)}(t)\Lambda^a_{\alpha(k)} \tag{101}$$

where $\Lambda^a_{\alpha(k)}$ contains operators of differentiation in time up to the order $(M - k - 1)$. Thus, one may make the following statement (which was in fact known to Noether[12]):

The conserved charge (98) which corresponds to any GT and its components $Q_{\alpha(k)}$ vanish on the equations of motion.

Let a global ST be the reduction of a GT to constant values of the parameters $\epsilon_\alpha(t)$. In this case the generators $\rho^a_{ga}(t)$ from equation (90) are just $\rho^a_{\alpha(0)}(t)$ from equation (93), and therefore $\delta q^a(t) = \rho^a_{\alpha(0)}(t)\epsilon_\alpha$. The corresponding conserved charges Q_α from (95) coincide with $Q_{\alpha(0)}$ from (98) and vanish on the equation of motion according to (101). The inverse statement is also valid, namely:

If some global continuous ST of an action $\delta q^a(t) = \rho^a(t)\epsilon$ generates a conserved charge which vanishes on the equation of motion, then this action obeys a gauge symmetry.

Let us prove this. Similar to (94), one can get

$$\frac{\partial L}{\partial q^a}\rho^a + \frac{\partial L}{\partial \dot{q}^a}\dot{\rho}^a = \frac{d}{dt}f \tag{102}$$

We can use this equation to write the following relation:

$$\frac{\partial L}{\partial q^a}\rho^a\epsilon(t)+\frac{\partial L}{\partial \dot{q}^a}\frac{d}{dt}[\rho^a\epsilon(t)]=\frac{d}{dt}[f\epsilon(t)]+\dot{\epsilon}(t)Q_{(0)} \tag{103}$$

where $Q_{(0)} = (\partial L/\partial \dot{q}^a)\rho^a - f$ is the conserved charge related to the global continuous ST [see (95)] and $\epsilon(t)$ is an arbitrary function of t. Let this charge vanish on the equations of motion, that is,

$$Q_{(0)}=\Lambda^a_{(0)}\frac{\delta S}{\delta q^a} \tag{104}$$

where $\Lambda^a_{(0)}$ may contain operators of differentiation with respect to time up to a finite order. Thus, the last term on the right-hand side of (103) has the form $\dot{\epsilon}(t)\Lambda^a_{(0)}\,\delta S/\delta q^a$. One can always write this term in the different form

$$\dot{\epsilon}(t)\Lambda^a_{(0)}\frac{\delta S}{\delta q^a}=-\frac{\delta S}{\delta q^a}\Lambda^a\epsilon(t)+\frac{d\varphi}{dt} \tag{105}$$

where Λ^a is an operator symmetric to $\Lambda^a_{(0)}$ and φ is some function. On the other hand,

$$\frac{\delta S}{\delta q^a}\Lambda^a\dot{\epsilon}(t)=\frac{\partial L}{\partial q^a}\Lambda^a\dot{\epsilon}(t)+\frac{\partial L}{\partial \dot{q}^a}\frac{d}{dt}[\Lambda^a\dot{\epsilon}(t)]-\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^a}\Lambda^a\dot{\epsilon}(t)\right] \tag{106}$$

Gathering (103), (105), and (106), we get

$$\delta L=\frac{\partial L}{\partial q^a}\delta q^a(t)+\frac{\partial L}{\partial \dot{q}^a}\delta\dot{q}^a(t)=\frac{d}{dt}\left[f\epsilon(t)+\varphi+\frac{\partial L}{\partial \dot{q}^a}\Lambda^a\dot{\epsilon}(t)\right]$$

where $\delta q^a(t)$ is a GT,

$$\delta q^a(t)=\rho^a\epsilon(t)+\Lambda^a\dot{\epsilon}(t) \tag{107}$$

Based on the two statements proved above we may define what can be called reparametrization ST in general. To this end let us first discover what is a global representative of such a symmetry. One can remember that in all known examples of finite-dimensional systems the existence of reparametrization invariance leads to the zero-Hamiltonian phenomenon. More exactly, the total Hamiltonian[4,5] appears to be proportional to constraints of the theory, or it vanishes on the equations of motion. Such a Hamiltonian can be derived from the expression for the Lagrangian energy if one replaces there all the primary expressible velocities as functions on phase space variables and denotes the primary inexpressible velocities by λ, which then play the role of Lagrange multipliers. Thus, in this case one can write

$$\mathscr{E} = \frac{\partial L}{\partial \dot{q}^a}\dot{q}^a - L = \Lambda^a_{(0)}\frac{\delta S}{\delta q^a} \tag{108}$$

Another observation is that in all known examples where RI takes place, the corresponding Lagrangians do not depend explicitly on time. Thus, we have the conservation law

$$\frac{d\mathscr{E}}{dt} = -\dot{q}^a\frac{\delta S}{\delta q^a} \tag{109}$$

On the other hand, one can interpret the energy $\mathscr{E}$ as a conserved charge related to the global ST, which are translations in time, $q^a(t) \to q^a(t+\epsilon)$, or in the infinitesimal form

$$\delta q^a(t) = \dot{q}^a(t)\epsilon \tag{110}$$

Indeed, in this case

$$\delta L = \frac{\partial L}{\partial q^a}\dot{q}^a\epsilon + \frac{\partial L}{\partial \dot{q}^a}\ddot{q}^a\epsilon = \epsilon\frac{dL}{dt} \tag{111}$$

so that (110) is a symmetry and, at the same time, (109) follows also from (111). Taking all this into account, it is natural to regard translations in time as global representatives of the reparametrization GT. Then one can define the latter GT as a possible extension of the translations in time to GT in the manner which was used in the proof of the inverse statement. Thus, such GT have the form (107) with $\rho^a = \dot{q}^a$,

$$\delta q^a(t) = \dot{q}^a(t)\epsilon(t) + \Lambda^a\dot{\epsilon}(t) \tag{112}$$

where the operators Λ^a are defined by the explicit form of the Lagrangian of the theory [see for example the transformations (2) and (4)].

Considering the above finite-dimensional case, we have seen that the conserved charge Q of (98) related to any GT and all its components $Q_{\alpha(\kappa)}$ vanish on the equations of motion. In particular, the components $Q_{\alpha(0)}$, which are the conserved charges related to the corresponding global ST (global representatives of the GT), with $\epsilon_\alpha(t) = \epsilon_\alpha = \text{const}$, also vanish on the equations of motion. However, such a conclusion may be wrong in the case of field theory.[13] As an example, let us take electrodynamics coupled to a scalar field $\varphi(x)$,

$$S = \int \mathscr{L} d^{D+1}x,$$

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu + ieA_\mu)\varphi^\dagger(\partial^\mu - ieA^\mu)\varphi - V(\varphi^\dagger\varphi) \tag{113}$$

The conserved charge [an analog of (95)], related to the GT $\delta A_\mu(x) = \partial_\mu \epsilon(x)$, $\delta\varphi(x) = ie\varphi(x)\epsilon(x)$, $\delta\varphi^\dagger(x) = -ie^\dagger(x)\epsilon(x)$, where $\epsilon(x)$ are parameters of the GT, is

$$Q = \int \left(\frac{\partial L}{\partial \dot{A}_\mu} \delta A_\mu + \frac{\partial L}{\partial \dot{\varphi}} \delta\varphi + \frac{\partial L}{\partial \dot{\varphi}^\dagger} \delta\varphi^\dagger \right) d^D x$$

$$= \int [F_{0k}\partial_k \epsilon(x) - j_0 \epsilon(x)]\, d^D x \tag{114}$$

$$j_0 = \varphi\dagger(\partial_0 - ieA_0)\varphi - \varphi(\partial_0 + ieA_0)\varphi^\dagger \tag{115}$$

This expression can be transformed by the equation of motion $\partial_k F_{0k} + iej_0 = 0$ to the form

$$Q = \int \partial_k [F_{0k}\epsilon(x)]\, d^D x \tag{116}$$

In the case of GT with $\epsilon(x)$ decreasing rapidly enough in the limit $|\mathbf{x}| \to \infty$, the charge (116) is zero. In the case of global ST with $\epsilon(t) = \epsilon = \text{const}$, we have

$$Q = \epsilon \int \partial_k F_{0k}\, d^D x = -ie\epsilon \int j_0\, d^D x \tag{117}$$

This expression may differ from zero. In the Coulomb phase F_{0k} behaves at large r as $r^{-(D-1)}$, so that the integral in (117) is proportional to the total electrical charge of the system, which is in general not zero. However, if a spontaneous symmetry breaking takes place (Higgs phase) the vector field becomes massive and F_{0k} decreases exponentially, resulting in $Q = 0$. (The total charge of any state is zero.)

One meets a similar situation in the theory of gravity. Let us select the action of the gravitational field of the form (first proposed by Dirac[14]; for a detailed treatment see refs. 9 and 5)

$$S = \int L d^4 x, \qquad L = A + \partial_i q^i \tag{118}$$

where

$$A = \sqrt{-g^{00} g_{(3)}} \left[\frac{z_{ik}}{4} (e^{il} e^{km} - e^{ik} e^{lm})\, z_{lm} - \frac{R_{(3)}}{g^{00}} \right],$$

$$g_{(3)} = |g_{ik}|, \qquad e^{ik} g_{kl} = \delta^i_l$$

$$z_{ik} = \dot{g}_{ik} - g_{0i,k} - g_{0k,i} + 2\gamma^l_{ik} g_{0l}, \qquad q^i = \sqrt{-g_{(3)}}\, g_{lm,k} (e^{il} e^{km} - e^{ik} e^{lm})$$

and γ^l_{ik} and $R_{(3)}$ are the Christoffel symbols and the scalar curvature constructed for the three-dimensional metric g_{ik}, respectively. This action is equivalent to the Einstein–Hilbert one under certain assumptions about the global structure of the theory. The Lagrangian L contains neither higher (second) order derivatives of the metric, nor velocities $\dot{g}_{0\mu}$. The variation of L under the GT (6) has the form $\delta L = \partial_\mu[L\epsilon^\mu(x)]$. The corresponding conserved charge is

$$Q = \int \left(\frac{\partial L}{\partial \dot{g}_{ik}} \delta g_{ik} - L\epsilon^0 \right) d^3x \tag{119}$$

If $\epsilon^\mu(x) \to 0$ when $|\mathbf{x}| \to \infty$, then one can see that it vanishes on the equations of motion. For example, if $\epsilon^i(x) \equiv 0$

$$Q = \int \epsilon^0 \left[g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} + g_{00} \frac{\delta S}{\delta g_{00}} \right] d^3x \tag{120}$$

In the case of $\epsilon_0(x) = \epsilon_0 = \text{const}$, $\epsilon_i(x) \equiv 0$, the charge (119) is proportional to the total energy and has the form

$$Q = -\epsilon_0 \int \partial_i q^i \, d^3x \tag{121}$$

The integral on the right-hand side of (121) is generally nonzero. In particular, in an asymptotically flat space[15] for a system with the total mass M

$$g_{ik} = -\delta^i_k \left(1 + \frac{M}{8\pi r} \right) + O\left(\frac{1}{r^2} \right) \tag{122}$$

Then $Q = \epsilon_0 M$ is not zero. One can remark, considering, for example, the theory of gravity, that in spite of the fact that four-dimensional divergence terms in the Lagrangian do not affect the form of the equations of motion, they can affect the form of the corresponding conserved charges. That may serve as an additional argument in favor of a certain form of the selected Lagrangian.

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